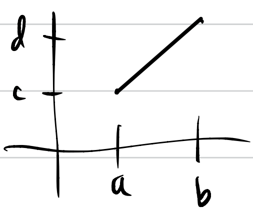
Metric Spaces and Topology
Lecture 10

A quick overview of basic set theory. Sets A and B are called *equinumerous* if $\exists$ bijection $A \xrightarrow{\sim} B$. We denote this by $A \equiv B$. We write $A \hookrightarrow B$ if $\exists$ injection $f: A \hookrightarrow B$ and we write $A \twoheadrightarrow B$ if there is a surjection $f: A \twoheadrightarrow B$.

Examples.

- **Hilbert hotel.** $\mathbb{N} \equiv \mathbb{N}^+$ by $n \mapsto n+1$.
- $[0,1] \cong [0,1) \cong (0,1)$. $\quad f: [0,1] \to [0,1)$
$$x \mapsto \begin{cases} \frac{1}{n+1} & x = \frac{1}{n} \\ x & \text{otherwise} \end{cases}$$
- $(a,b) \equiv (c,d)$. $\quad F: (a,b) \to (c,d)$
$$x \mapsto \frac{d-c}{b-a}(x-a) + c.$$

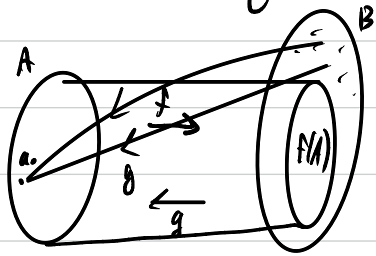


- $\mathbb{R} \equiv (a,b) \equiv (0,1)$ $\quad f: (-1,1) \to \mathbb{R}$
$$x \mapsto \frac{x}{1-|x|}.$$
$\boxed{\text{HW}}$ Verify that this is a bijection.

Proposition. For sets $A, B \neq \emptyset$, $A \hookrightarrow B \overset{AC}{\iff} B \twoheadrightarrow A$.

Proof. $\Rightarrow$. $f: A \hookrightarrow B$, fix $a_0 \in A$ and define

$$g: B \to A \text{ by } b \mapsto \begin{cases} a & \text{if } b = f(a) \text{ for some } a \in A \\ a_0 & \text{otherwise} \end{cases}.$$

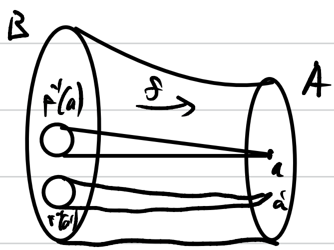


$\Leftarrow$. $f: B \twoheadrightarrow A$. By the Axiom of Choice, there is a function $c: P(B) \setminus \{\emptyset\} \to B$

$$B' \mapsto b \in B'.$$

Define $g: A \to B$ by

$$a \mapsto c(f^{-1}(a)).$$ $\square$

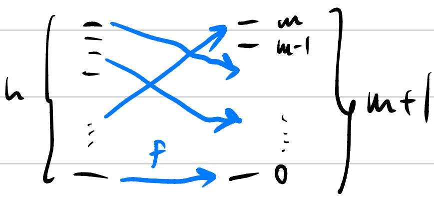


Def. A set is called *finite* if it is equinumerous to a natural number (where we think of $n \in \mathbb{N}$ as the set $n := \{0, 1, \dots, n-1\}$). Otherwise, we say that it is *infinite*. $n < m$

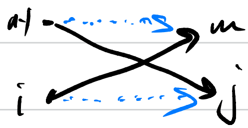
Def. A set is called *Dedekind infinite* if it is equinumerous with its proper subset. Otherwise we call it Dedekind finite.

Prop (Pigeonhole Principle).
Finite sets are Dedekind finite. In fact, if $n \hookrightarrow m$ then $n \leq m$, for all $n, m \in \mathbb{N}$.

Proof. We do induction on m. If m is $0 = \emptyset$, then if $n \neq \emptyset$, then $\nexists$ a function $n \to m$, thus $n = \emptyset$. For the step, suppose the statement is true for m and prove it for $m+1$. If $m \notin f(\{0, \dots, n-1\})$, then



$f : n \to m := \{0, \dots, m-1\}$, so by induction $n \leq m$.
Otherwise,



We change the function f so that $f(n-1) = m$.
But then $f|_{\overbrace{\{0,\dots,n-2\}}^{n-1}} \hookrightarrow m := \{0, 1, \dots, m-1\}$, so by induction, $n-1 \leq m$, hence $n \leq m+1$. $\square$

Theorem. For a set A, TFAE:
(1) A is infinite.
(2) A is Dedekind infinite.
(3) $\mathbb{N} \hookrightarrow A$.
(4) $A \twoheadrightarrow \mathbb{N}$.

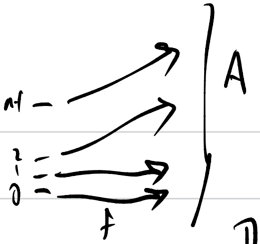
Proof. (2) => (1). This just Pigeonhole Principle.
(1) => (3). By Axiom of Choice $\boxed{\text{HW}}$
(3) <=> (4). Already done.
(3) => (2). Hilbert hotel.

$\boxed{\text{HW}}$ Prove directly that (2) => (3) without AC.

Def. A set A is called countable if $A \approx \mathbb{N}$ or A is finite.

Prop. For a set A, the following are equivalent:
(1) A is ctbl (= countable).
(2) $A \hookrightarrow \mathbb{N}$.
(3) $\mathbb{N} \twoheadrightarrow A$.
↑ no AC is needed

Proof. (1) => (2). Trivial.
(2) <=> (3). Already done, uses AC.
(2) => (1). Without loss of gen., assume $A \subseteq \mathbb{N}$.
Suppose A is infinite and we build $f : \mathbb{N} \to A$ recursively as follows:
Let $n \in \mathbb{N}$ and suppose $f|_{n = \{0, \dots, n-1\}}$ is



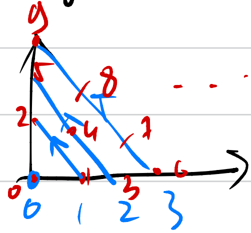
already defined and define $f(n) :=$ least element in $A \setminus \{f(0), f(1), \dots, f(n-1)\}$.

This is easily a bijection. $\square$

Examples.

∘ $\mathbb{N} \times \mathbb{N}$

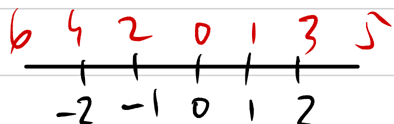
$\mathbb{N} \cong \mathbb{N}^2 \cong \mathbb{N}^2 \times \mathbb{N} = \mathbb{N}^3 \cong \mathbb{N}^k$.



$f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$(n, m) \mapsto \frac{(n+m)(n+m+1)}{2} + n$.

∘ $\mathbb{Z}$ is ctbl



$\mathbb{N} \cong \mathbb{N} \times \mathbb{N}$

$\mathbb{Q}$ is ctbl because $\mathbb{Z} \times \mathbb{N}^+ \twoheadrightarrow \mathbb{Q}$.

$(n, m) \mapsto \frac{n}{m}$

If A is ctbl then so is $A^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} A^n$.

$\boxed{\text{HW}}$ Prove without AC.

Prop (AC). A ctbl union of ctbl sets is ctbl, i.e. if $\mathcal{A}$ is a ctbl set whose each element is ctbl, then $\bigcup \mathcal{A} := \bigcup_{A \in \mathcal{A}} A$ is also ctbl.

Proof. We use the $\mathbb{N} \twoheadrightarrow$ definition of ctbility.

Let $f: \mathbb{N} \twoheadrightarrow \mathcal{A}$ be a surjection, i.e. $\mathcal{A} = (A_n)_{n \in \mathbb{N}}$. We know that for each n $\exists$ a surjection $\mathbb{N} \twoheadrightarrow A_n$. We use AC to get a function $F: \mathbb{N} \to (\bigcup_n A_n)^{\mathbb{N}}$ $n \mapsto F_n$, where F_n is a surjection $\mathbb{N} \twoheadrightarrow A_n$. Thus, we can build a surjection $\mathbb{N} \cong \mathbb{N} \times \mathbb{N} \to \bigcup_n A_n$ by $(n,m) \mapsto F_n(m)$. $\square$

HW Prove that the set of algebraic numbers is ctbl. A real number r is called algebraic if it is a root of a polynomial with rational coefficients, e.g. $\sqrt{2}$ is algebraic being a root of $x^2 - 2$. All rational numbers are algebraic because $q \in \mathbb{Q}$ is a root of $x - q$.